Resonance for Quasilinear Elliptic Higher Order Partial Differential Equations at the First Eigenvalue

by

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ABSTRACT

We present a resonance result on the Sobolev space $W^{m,p}(\Omega)$, where Ω is a bounded open connected subset of R^N meeting the cone property. We let $1 < p < \infty$ and Qu be the $2m^{th}$ order quasilinear differential operator in generalized divergence form

$$Qu = \sum_{1 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi_m(u)),$$

for $u \in W^{m,p}$, where $\xi_m = \{D^\alpha u : 0 \leq |\alpha| \leq m\}$, and we make standard assumptions on A_α such as Carathéodory, uniform ellipticity, monotonicity, and a growth restriction. We study an equation of the following nature,

$$Qu(x) = g(x, u(x)) + h(x), \text{ for } u \in W^{m,p}(\Omega),$$

where $h(x) \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, and $g(x,t) : \Omega \times R \to R$ is Carathéodory. Subject to $mp > N$, we show the existence of a solution to the above equation with g having superlinear growth in u but subject to a one-sided growth condition.

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1. Introduction. In this paper, the author presents a resonance result on the Sobolev space $W^{m,p}(\Omega)$, where Ω is a bounded open connected subset of R^N meeting the cone property. We let $1 < p < \infty$ and Qu be the $2m^{th}$ order quasilinear differential operator in generalized divergence form

$$Qu = \sum_{1 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \xi_m(u)), \tag{1.1}$$

for $u \in W^{m,p}$, where $\xi_m = \{D^\alpha u : 0 \leq |\alpha| \leq m\}$, and we make standard assumptions on A_α such as Carathéodory, uniform ellipticity, monotonicity, and a growth restriction. We shall study an equation of the following nature,

$$Qu(x) = g(x, u(x)) + h(x), \text{ for } u \in W^{m,p}(\Omega), \tag{1.2}$$

where $h(x) \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, and $g(x,t) : \Omega \times R \to R$ is Carathéodory. Subject to $mp > N$, we show the existence of a solution to (1.2) with g having superlinear growth in u but subject to a one-sided growth condition. Since Q lacks an $\alpha = 0$ order term, problem (1.2) is considered at resonance since $Qu = \lambda_1 u$ is solved by $\lambda_1 = 0$ and u =constant, where λ_1 is defined as the first eigenvalue of Q. Shapiro[9, p. 365] provides a detailed explanation of this. This result primarily differs from that of Shapiro[9] in that our one-sided growth assumption on g is different from his, and since we approached the first eigenvalue of Q from values bigger than $\lambda_1 = 0$, in order for our results to hold, our Landesman-Lazer conditions must have reversed inequalities from those of Shapiro's Theorem 1 [9, p. 365]. Thus the theorem we will establish in this paper, holds for a distinct class of functions that those meeting the hypothesis of Shapiro's Theorem 1. Examples meeting our conditions on g, but not covered by Shapiro[9], will be provided in the next section. However, we do point out that Shapiro[9] takes $h \in (W^{m,p})^*$, the dual of $W^{m,p}$, and that while his superlinear growth condition on g holds for a general p, its growth is governed by q where if $p < Nm^{-1}$ then $q = \frac{pN}{N-mp}$ and $q' = \frac{q}{q-1}$ for $q > p$. Thus his results in this sense are more general.

2. Preliminaries. In this section, we introduce the necessary notation and establish preliminary results in order to prove the theorems in the following sections. We begin by letting $\Omega \subset R^N, N \geq 1$, be a bounded open connected set meeting the cone property, i.e., there exists a finite cone C such that each point x in Ω is a vertex of a finite cone C_x contained in Ω and congruent to C (see [2, page 11] or [1, page 66]). Thus, in particular, Ω cannot contain any cusps. The points of the open set Ω will be designated by $x = (x_1, ..., x_N)$, and the elementary differential operators by $D^\alpha = \prod_{j=1}^N (\partial/\partial x_j)^{\alpha_j}$ for an ordered N-tuple $\alpha = (\alpha_1, ..., \alpha_N)$ of nonnegative integers with the order of the operator D^α being written as $|\alpha| = \sum_{j=1}^N \alpha_j$. To write nonlinear partial differential operators in a convenient form, we introduce the vector space R^{s_m} whose elements are $\xi_m = \{\xi_\alpha : |\alpha| \leq m\}$ and divide each ξ_m into two parts $\xi_m = (\eta_{m-1}, \zeta_m)$ where $\eta_{m-1} = \{\eta_\beta : |\beta| \leq m-1\} \in R^{s_{m-1}}$ is the lower order part of ξ_m and $\zeta_m = \{\zeta_\alpha : |\alpha| = m\}$ is the part corresponding to the m^{th} derivatives (i.e., the highest order terms). For $u \in W^{m,p}(\Omega)$, $\xi_m(u)(x) = \{D^\alpha u(x) :$

$|\alpha| \le m\}$. (Note, $D^{(0,0,\ldots,0)}u = u$.) Furthermore, the semilinear form of the operator given by (1.1) is

$$Q(u,v) = \sum_{1 \le |\alpha| \le m} \int_\Omega A_\alpha(x, \xi_m(u)) D^\alpha v, \quad \forall u, v \in W^{m,p}(\Omega). \tag{2.1}$$

We make the following usual assumptions on the coefficients of Q.

(A-1) Each $A_\alpha : (\Omega \times R^{s_m}) \to R$ satisfies the Carathéodory conditions, i.e., $A_\alpha(x, \xi_m)$ is measurable for x in Ω for every fixed $\xi_m \in R^{s_m}$ and continuous in ξ_m for a.e. fixed $x \in \Omega$.

(A-2) There exist constants p with $1 < p < \infty$, $c \ge 0$, and a nonnegative function $\tilde{h} \in L^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ such that: $|A_\alpha(x, \xi_m)| \le \tilde{h}(x) + c|\xi_m|^{p-1}$, $1 \le |\alpha| \le m$ for a.e. $x \in \Omega$, for all $\xi_m \in R^{s_m}$.

(A-3) $\sum_{|\alpha|=m} (A_\alpha(x, \eta_{m-1}, \zeta_m) - A_\alpha(x, \eta_{m-1}, \zeta_m'))(\zeta_\alpha - \zeta_\alpha') > 0$ for a.e. $x \in \Omega$, for all $(\eta_{m-1}, \zeta_m) \in R^{s_m}, \zeta_m \ne \zeta_m'$ where $A_\alpha(x, \xi_m) = A_\alpha(x, \eta_{m-1}, \zeta_m)$ with $\xi_m = (\eta_{m-1}, \zeta_m)$.

This is known as the monotonicity condition which will be needed when establishing results for $|\alpha| = m$.

(A-4) There exist a positive constant $c_o > 0$ such that

$$\sum_{1 \le |\alpha| \le m} A_\alpha(x, \xi_m) \xi_\alpha \ge c_o \{ \sum_{1 \le |\alpha| \le m} |\xi_\alpha|^2 \}^{p/2},$$

for a.e. $x \in \Omega$, for all $\xi_m \in R^{s_m}$ and p is as given in (A-2).

This is known as the uniform ellipticity condition.

Moreover, we make the following assumptions on $g(x,t)$.

(g-1) $g(x,t)$ meets the usual Carathéodory conditions.

(g-2) $g(x,t)$ grows superlinearly, that is,

for all $\epsilon > 0$, there exist a $g_\epsilon \in L^{p'}(\Omega)$ such that

$|g(x,t)| \leq \epsilon |t|^{p-1} + g_\epsilon(x), \quad g_\epsilon(x) \geq 0, \qquad$ a.e.$x \in \Omega$,

for all $t \in R$, and $mp > N$.

(g-3) $g(x,t)$ meets the following one-sided growth condition,

$tg(x,t) \geq -c(x)|t| - d(x), \quad c(x), d(x) \geq 0$ a.e. $x \in \Omega$

and in $L^{p'}(\Omega)$, for all $t \in R$.

Before providing examples of functions meeting (g-1)-(g-3), we state the main theorem we will establish in this paper.

Theorem 2.1 *Let $mp > N$ and let $\Omega \subset R^N$ be a bounded domain with the cone property. Suppose g meets (g-1)-(g-3), $h \in L^{p'}$, and Qu is given by (1.1) where $A_\alpha(x, \xi_m)$ satisfies (A-1)-(A-4) for $1 \leq |\alpha| \leq m$, and we set*

$$g_-(x) = \limsup_{t \to -\infty} g(x,t) \quad \textit{and} \quad g_+(x) = \liminf_{t \to +\infty} g(x,t).$$

Furthermore, suppose the following type of Landesman-Lazer condition prevails,

(2.2) $$\int_\Omega g_-(x) < -\int_\Omega h(x) < \int_\Omega g_+(x),$$

then (1.2) has a weak solution.

By a *weak solution* we mean that there exists a $u \in W^{m,p}(\Omega)$ such that,

(2.3) $$Q(u,v) = \int_\Omega g(x,u)v + \int_\Omega hv, \quad \forall v \in W^{m,p}(\Omega),$$

where $Q(u,v)$ is given by (2.1). Examples of functions satisfying the hypothesis of Theorem 2.1 but not meeting those of Theorem 1 in Shapiro[9, p. 365], are

Example 2.2 Let N=1, $\Omega = (0, 2\pi)$, and

$$g(x,t) = \begin{cases} |sinx|\frac{t^{p-1}}{\log t}, & \text{for } t \geq 2, \\ |sinx|\frac{2^{p-1}}{\log 2}(t-1), & \text{for } 1 \leq t \leq 2, \\ 0, & \text{for } 0 \leq t \leq 1, \\ -g(x,-t), & \text{for } t < 0. \end{cases}$$

Also consider

Example 2.3 Let N=1, $\Omega = (0, 2\pi)$, and

$$g(x,t) = \begin{cases} |cosx|t^{p-1-\epsilon}, & \text{for } t \geq 0 \text{ and for } p > 1+\epsilon, \text{ where } 0 < \epsilon < 1, \\ -g(x,-t), & \text{for } t < 0. \end{cases}$$

It is straight forward to verify that $g(x,t)$, in both illustrations, is an odd (in t) continuous function that meets conditions (g-1)-(g-3). In particular, for both of these cases, we have that $g_-(x) = \lim_{t\to-\infty} g(x,t) = -\infty$ and $g_+(x) = \lim_{t\to\infty} g(x,t) = +\infty$. Hence, the Landesman-Lazer conditions (2.2) are certainly met, but not those conditions of Theorem 1 appearing in Shapiro [9]. He imposes conditions which would necessitate the existence of $h \in L^{p'}$ so that $+\infty < -\int_\Omega h(x) < -\infty$ which is absurd. The reversal in the inequalities in the Landesman-Lazer conditions occurred because, in order to establish his results, Shapiro[9] required that $g_+ = \limsup_{t\to\infty} g(x,t)$ and that $g_- = \liminf_{t\to-\infty} g(x,t)$. On a final note, it is an easy matter to verify that our illustrations also do not meet his one-sided growth condition which is that $g(x,t)t \leq q(x)|t|$ for a.e. $x \in \Omega$ and for all $t \in R$, for some $q(x) \geq 0$ for a.e. $x \in \Omega$ and in $L^{p'}$.

For the proof of the theorem, we need the following fact established in Shapiro [8, pages 1852-1854]. If $1 < p < \infty$ and Ω is a bounded open connected set with the cone property, then there exist a sequence $\{\phi_n\}_{n=1}^{\infty}$ in $W^{m,p}(\Omega)$ such that the following properties hold:

$$\{\phi_n\}_{n=1}^{\infty} \text{ is a Complete Orthonormal System (CONS) in } L^2(\Omega);$$

$$\phi_1(x) = |\Omega|^{-\frac{1}{2}};$$

$$\phi_n \in W^{m,2} \cap W^{m,p} \text{ for } n = 1, 2, ... \tag{2.4}$$

Furthermore, from Shapiro [8, pages 1852-1854] we see that if we let

$$S_J = \text{subspace of } W^{m,p}(\Omega) \text{ spanned by } \{\phi_1, \phi_2, ..., \phi_J\}, \tag{2.5}$$

then given $v \in W^{m,p}(\Omega), \exists \{v_J\} \in S_J$ such that

$$\lim_{J \to \infty} \|v - v_J\|_{W^{m,p}} = 0. \tag{2.6}$$

We next define

$$g^n(x,t) = \begin{cases} n, & \text{if } g(x,t) \geq n, \\ g(x,t), & \text{if } |g(x,t)| \leq n, \\ -n, & \text{if } g(x,t) \leq -n. \end{cases} \tag{2.7}$$

Following the Galerkin method, (see Kesavan [4]), the theorem is proved by first showing that a solution, say u_J, exists for the following perturbed problem which is a nonresonance result in the finite dimensional space S_J. This proposition will be invoked when establishing results on $W^{m,p}(\Omega)$.

Proposition 2.4 *Let n be a fixed positive integer. Under the hypothesis of Theorem 2.1, we will show that there exists a weak solution, $u_J \in S_J$, of*

$$Qu - \frac{1}{n} sgn(u)|u|^{p-1} = g^n(x,u) + h(x), \quad u \in S_J. \tag{2.8}$$

Observe that for n a fixed large positive integer, $g^n(x,t)$ is bounded by n. Consequently, we are not assuming superlinear growth in establishing Proposition 2.4.

Thus, by a *weak solution* we mean a $u_J \in S_J$ such that

$$(2.9)\quad Q(u_J,v) - \frac{1}{n}\int_\Omega sgn(u_J)|u_J|^{p-1}v = \int_\Omega g^n(x,u_J)v + \int_\Omega hv, \quad \forall v \in S_J,$$

where

$$\mathrm{sgn}(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -1, & \text{if } t < 0. \end{cases}$$

Proof of Proposition 2.4.

To establish the proposition, define for $\beta = (\beta_1^J, \beta_2^J, ..., \beta_J^J) \in R^J$ the following,

$$\begin{aligned}
F_1(\beta) &= -Q(\sum_{j=1}^J \beta_j^J\phi_j, \phi_1) \\
&+ \frac{1}{n}\int_\Omega \mathrm{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}\phi_1 \\
&+ \int_\Omega g^n(x, \sum_{j=1}^J \beta_j^J\phi_j)\phi_1 + \int_\Omega h\phi_1 \\
(F_k(\beta))_{k=2}^J &= Q(\sum_{j=1}^J \beta_j^J\phi_j, \phi_k) \\
&- \frac{1}{n}\int_\Omega \mathrm{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}\phi_k \\
(2.10)\qquad &- \int_\Omega g^n(x, \sum_{j=1}^J \beta_j^J\phi_j)\phi_k - \int_\Omega h\phi_k.
\end{aligned}$$

Setting $F(\beta) = (F_1(\beta), ..., F_J(\beta))$, multiplying both sides of (2.10) by β_k^J, summing on k, using the fact that ϕ_1 is a constant (see (2.4)), and applying

(2.1) we have

$$
\begin{aligned}
(F(\beta)\cdot\beta) &= \frac{1}{n}\int_\Omega \operatorname{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}\beta_1^J\phi_1 \\
&+ \int_\Omega g^n(x,\sum_{j=1}^J \beta_j^J\phi_j)\beta_1^J\phi_1 + \int_\Omega h\beta_1^J\phi_1 \\
&+ Q(\sum_{j=1}^J \beta_j^J\phi_j, \sum_{k=2}^J \beta_k^J\phi_k) \\
&- \frac{1}{n}\int_\Omega \operatorname{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}\sum_{k=2}^J \beta_k^J\phi_k \\
&- \int_\Omega g^n(x,\sum_{j=1}^J \beta_j^J\phi_j)(\sum_{k=2}^J \beta_k^J\phi_k) \\
&- \int_\Omega h(\sum_{k=2}^J \beta_k^J\phi_k). \qquad (2.11)
\end{aligned}
$$

Note: For the remainder of this paper, we will be using the L^p-norm unless otherwise indicated.

Moreover, since Q is linear on the second variable (see (2.1)), applying Cauchy-Schwarz's inequality, (A-4), the definition of g^n, and the following equality,

$$
-\sum_{j=1}^J \beta_j^J\phi_j + 2\sum_{k=2}^J \beta_k^J\phi_k = -\beta_1^J\phi_1 + \sum_{k=2}^J \beta_k^J\phi_k,
$$

we obtain

$$
\begin{aligned}
(F(\beta)\cdot\beta) &= Q(\sum_{j=1}^J \beta_j^J\phi_j, \sum_{k=2}^J \beta_k^J\phi_k) \\
&- \frac{1}{n}\int_\Omega \operatorname{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}(-\beta_1^J\phi_1 + \sum_{k=2}^J \beta_k^J\phi_k) \\
&- \int_\Omega g^n(x,\sum_{j=1}^J \beta_j^J\phi_j)(-\beta_1^J\phi_1 + \sum_{k=2}^J \beta_k^J\phi_k) - \int_\Omega h(-\beta_1^J\phi_1 + \sum_{k=2}^J \beta_k^J\phi_k) \\
&\geq c_o\int_\Omega\{\sum_{1\leq|\alpha|\leq m} |D^\alpha \sum_{k=2}^J \beta_k^J\phi_k|^2\}^{p/2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{n}\int_\Omega \operatorname{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}(-\sum_{j=1}^J \beta_j^J\phi_j + 2\sum_{k=2}^J \beta_k^J\phi_k) \\
&- \|n\|_{p'}\|\sum_{k=2}^J \beta_k^J\phi_k\| - \|h\|_{p'}\|\sum_{k=2}^J \beta_k^J\phi_k\| \\
&- \|n\|_{p'}\|\beta_1^J\phi_1\| - \|h\|_{p'}\|\beta_1^J\phi_1\|. \qquad (2.12)
\end{aligned}
$$

Inequality (2.12) reduces to

$$
\begin{aligned}
(F(\beta)\cdot\beta) \;\geq\; & c_o\int_\Omega\{\sum_{1\leq|\alpha|\leq m}|D^\alpha\sum_{k=2}^J \beta_k^J\phi_k|^2\}^{p/2} \\
&+ \frac{1}{n}\int_\Omega \operatorname{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}(\sum_{j=1}^J \beta_j^J\phi_j) \\
&- \frac{2}{n}\int_\Omega \operatorname{sgn}(\sum_{j=1}^J \beta_j^J\phi_j)|\sum_{j=1}^J \beta_j^J\phi_j|^{p-1}(\sum_{k=2}^J \beta_k^J\phi_k) \\
&- (\|n\|_{p'} + \|h\|_{p'})\|\sum_{k=2}^J \beta_k^J\phi_k\| \\
&- (\|n\|_{p'} + \|h\|_{p'})\|\beta_1^J\phi_1\|. \qquad (2.13)
\end{aligned}
$$

Using the fact that $\int_\Omega(\sum_{k=2}^J \beta_k^J\phi_k)\cdot\phi_1 = 0$, from the generalized Poincare's inequality (see [5, page 32]), we have that there exists a positive constant $k_1 = K_1(\Omega, p) > 0$ such that

$$
c_o\int_\Omega\{\sum_{1\leq|\alpha|\leq m}|D^\alpha\sum_{k=2}^J \beta_k^J\phi_k|^2\}^{p/2} \geq c_o k_1\int_\Omega|\sum_{k=2}^J \beta_k^J\phi_k|^p. \qquad (2.14)
$$

Letting $\delta = c_o k_1$, applying (2.14) to (2.13), and using Hölder's inequality we have

$$
\begin{aligned}
(F(\beta)\cdot\beta) \;\geq\; & \delta\int_\Omega|\sum_{k=2}^J \beta_k^J\phi_k|^p + \frac{1}{n}\int_\Omega|\sum_{j=1}^J \beta_j^J\phi_j|^p \\
&- \frac{2}{n}\|\sum_{j=1}^J \beta_j^J\phi_j\|^{p-1}\|\sum_{k=2}^J \beta_k^J\phi_k\| \\
&- (\|n\|_{p'} + \|h\|_{p'})\|\sum_{k=2}^J \beta_k^J\phi_k\| \\
&- (\|n\|_{p'} + \|h\|_{p'})\|\beta_1^J\phi_1\|. \qquad (2.15)
\end{aligned}
$$

By Young's inequality (see [6]), one can show that, for n chosen big enough since $\delta > 0$ and $p > 1$, we have

$$(2.16) \qquad \delta \|\sum_{k=2}^{J} \beta_k^J \phi_k\|^p + \frac{1}{n}\|\sum_{j=1}^{J} \beta_j^J \phi_j\|^p \geq \frac{3}{n}\|\sum_{j=1}^{J} \beta_j^J \phi_j\|^{p-1}\|\sum_{k=2}^{J} \beta_k^J \phi_k\|.$$

Inequality (2.16) will follow as a consequence of the following claim.

Claim 1 *Let $\delta > 0$, $p > 1$, and $p' = \frac{p}{p-1}$. Then $\exists n_o$ s.t. for $n \geq n_0$,*

$$(2.17) \qquad \delta \|\sum_{k=2}^{J} \beta_k^J \phi_k\|^p + \frac{1}{2n}\|\sum_{j=1}^{J} \beta_j^J \phi_j\|^p \geq \frac{3}{n}\|\sum_{j=1}^{J} \beta_j^J \phi_j\|^{p-1}\|\sum_{k=2}^{J} \beta_k^J \phi_k\|.$$

Proof of Claim 1:

For simplicity of notation let $A = \|\sum_{j=2}^{J} \beta_j^J \phi_j\|$ and $B = \|\sum_{j=1}^{J} \beta_j^J \phi_j\|^{p-1}$ then $B^{p'} = \|\sum_{j=1}^{J} \beta_j^J \phi_j\|^p$. Substituting these values in (2.17) and multiplying both sides by $\frac{n}{3}$, we see that (2.17) prevails if and only if the following holds.

$$(2.18) \qquad \frac{n}{3}\delta A^p + \frac{p'}{6}\frac{B^{p'}}{p'} \geq AB.$$

However, (2.18) holds if and only if the following does.

$$\frac{6n\delta}{3p'}A^p + \frac{B^{p'}}{p'} \geq \frac{6}{p'}AB.$$

Setting $C = \frac{6}{p'}A$ gives $A^p = (\frac{p'}{6})^p C^p$. Thus the claim holds if and only if

$$(2.19) \qquad I_n = \frac{6n\delta}{3p'}(\frac{p'}{6})^p C^p + \frac{B^{p'}}{p'} \geq CB$$

is true. However, for n chosen large enough, it is the case that

$$\frac{6n\delta}{3p'}(\frac{p'}{6})^p \geq \frac{1}{p}.$$

Thus, using the above inequality, we see that (2.19) holds if and only if

$$I_n \geq \frac{C^p}{p} + \frac{B^{p'}}{p'} \geq CB.$$

But, this is Young's inequality (see [6]). Therefore, Claim 1 is established. Next, with $|\beta|^2 = (\beta_1^J)^2 + ... + (\beta_J^J)^2$, from (2.4) and since $mp > N$, by the Rellich-Kondrachov theorem[1, p. 144], we have that $\phi_i \in L^{p'}(\Omega)\forall i$, thus it follows that

$$\lim_{|\beta|\to\infty} \|\sum_{j=1}^{J} \beta_j^J \phi_j\|_p = \infty.$$

Applying (2.16) to (2.15), since $\delta > 0$, $n > 0$, and $p > 1$ then $(F(\beta) \cdot \beta) \to \infty$ as $|\beta| \to \infty$. Hence, there exists a $\rho > 0$ such that

$$(F(\beta) \cdot \beta) > 0 \quad \text{for} \quad |\beta| \geq \rho. \tag{2.20}$$

However, in order to apply the corollary to Brouwer's fixed point theorem (see Kesavan [4, page 219]), we need to show that $F_i(\beta) \in C(R^J, R)$ for $i = 1, 2, ..., J$. This follows from the definition of each $F_i(\beta)$, from (A-1), (A-2), (g-1), and since g^n is bounded. Therefore,

$$F_i(\beta) \in C(R^J, R), \text{ for } i = 1, 2, ..., J.$$

Thus, we have that there exist $|\hat{\beta}| \leq \rho$, $\hat{\beta} \in R^J$, such that

$$F_i(\hat{\beta}) = 0 \quad \text{for all} \quad i = 1, 2, ..., J.$$

Set $u_J = \sum_{i=1}^{J} \hat{\beta}_i^J \phi_i$ and observe from (2.10) that

$$Q(u_J, \phi_k) - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}\phi_k = \int_\Omega g^n(x, u_J)\phi_k + \int_\Omega h\phi_k, \quad \text{for } k = 1, 2, ..., J.$$

This gives (2.9) and the proof of Proposition 2.4 is complete. ■

3. Nonresonance $W^{m,p}(\Omega)$. Proceeding along with the Galerkin approximation argument. By invoking Proposition 2.4 for each J, we will be able to obtain a sequence of solutions, u_J, which we will show to be uniformly bounded independent of J in $W^{m,p}$. Furthermore, this sequence will have a weak limit which will converge to a solution of the following proposition. This proposition, that we establish next, is a nonresonance result in the Sobolev Space $W^{m,p}(\Omega)$.

Proposition 3.5 *Let n be a fixed positive integer. Under the hypothesis of Proposition 2.4, we will show there exist $u_n \in W^{m,p}(\Omega)$ such that*

$$(3.1) \qquad Q(u_n, v) - \frac{1}{n}\int_\Omega sgn(u_n)|u_n|^{p-1}v = \int_\Omega g^n(x, u_n)v + \int_\Omega hv,$$

for all $v \in W^{m,p}(\Omega)$.

Proof of Proposition 3.5:

Since n is a fixed positive integer, we invoke Proposition 2.4. This gives us a sequence $\{u_J\}_{J=1}^\infty$ such that $u_J \in S_J$ satisfies (2.9) for $J = 1, 2, ...$ Before we proceed with the proof, we show the following needed claim.

Claim 2 *The sequence*

$$\{||u_J||_{W^{m,p}}\}_{J=1}^\infty \quad \textit{is uniformly bounded.}$$

Proof of Claim 2:

Suppose the claim is false. Then, it suffices to assume that

$$(3.2) \qquad \{||u_J||_{L^p}\}_{J=1}^\infty \to \infty \quad \text{as} \quad J \to \infty.$$

For if $||u_J||_{L^p}$ is uniformly bounded, then we are done by the following argument. Take $v = u_J$ in (2.9) and apply (A-4) to obtain,

$$c_o \int_\Omega \{ \sum_{1\le|\alpha|\le m} |D^\alpha u_J|^2\}^{\frac{p}{2}} \ \le \ Q(u_J, u_J)$$

$$
\begin{aligned}
&= \frac{1}{n}||u_J||^p + \int_\Omega g^n(x, u_J)u_J + \int_\Omega h u_J \\
&\leq \frac{1}{n}||u_J||^p + ||n||_{p'}||u_J|| + ||h||_{p'}||u_J|| \\
&\leq k,
\end{aligned}
$$

for some $k > 0$ since n is fixed. Next, since $p > 1$, we see that there exists a constant $\delta > 0$ such that

$$
\begin{aligned}
c_o \int_\Omega \{ \sum_{1 \leq |\alpha| \leq m} |D^\alpha u_J|^2 \}^{\frac{p}{2}} &\geq \delta c_o \int_\Omega \sum_{1 \leq |\alpha| \leq m} |D^\alpha u_J|^p \\
&= \delta c_o \sum_{1 \leq |\alpha| \leq m} ||D^\alpha u_J||^p, \quad \forall n.
\end{aligned}
$$

The above two inequalities imply that

$$\sum_{1 \leq |\alpha| \leq m} ||D^\alpha u_J||^p \leq k, \text{ for some } k > 0.$$

But this together with the assumption that the L^p-norm of this sequence is bounded gives

$$||u_J||_{W^{m,p}} \leq k, \quad \text{for some} \quad k > 0.$$

Thus establishing Claim (2). Consequently, we continue under the assumption that (3.2) holds.

For simplicity of notation, we let

$$\tilde{u}_J = -\beta_1^J \phi_1 + \sum_{j=2}^J \beta_j^J \phi_j,$$

where

$$u_J = \sum_{j=1}^J \beta_j^J \phi_j \tag{3.3}$$

and

$$u_{J2} = \sum_{j=2}^J \beta_j^J \phi_j.$$

Then, from (3.3), it follows that $\tilde{u}_J = -u_J + 2u_{J2}$. Thus taking $v = \tilde{u}_J$ in (2.9), using (3.3) and (2.7), we obtain

$$
\begin{aligned}
Q(u_J, \tilde{u}_J) &= \frac{1}{n}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(-u_J + 2u_{J2}) \\
&+ \int_\Omega g^n(x, u_J)\tilde{u}_J + \int_\Omega h\tilde{u}_J \\
&= -\frac{1}{n}\int_\Omega |u_J|^p + \frac{2}{n}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(u_{J2}) \\
&+ \int_\Omega g^n(x, u_J)\tilde{u}_J + \int_\Omega h\tilde{u}_J \\
&\leq -\frac{1}{n}||u_J||^p + \frac{2}{n}||u_J||^{p-1}||u_{J2}|| \\
&+ ||n||_{p'}||\tilde{u}_J|| + ||h||_{p'}||\tilde{u}_J||. \qquad (3.4)
\end{aligned}
$$

Next, recall that by Poincare's inequality, (A-4), and (2.1) we have that,

$$\exists\delta > 0 \text{ such that } Q(u_J, \tilde{u}_J) \geq \delta||u_{J2}||^p. \qquad (3.5)$$

Applying (3.5) to (3.4) we have

$$\delta||u_{J2}||^p \leq -\frac{1}{n}||u_J||^p + \frac{2}{n}||u_J||^{p-1}||u_{J2}|| + (||n||_{p'} + ||h||_{p'})||\tilde{u}_J||. \qquad (3.6)$$

Now, since

$$|\beta_1^J\phi_1| = |\hat{u}_J(1)\phi_1| = |\frac{\phi_1}{|\Omega|^{\frac{1}{2}}}\int_\Omega u_J| \leq \frac{|\Omega|^{\frac{1}{p'}}}{|\Omega|}||u_J||$$

for $u_{J2} = u_J - \beta_1^J\phi_1$, then for some $k > 0$ $||u_{J2}|| \leq k||u_J||$. Consequently we have that $||\tilde{u}_J|| \leq k'||u_J||$, for some $k' > 0$. Applying this to (3.6) and moving terms to the left-hand side we obtain,

$$
\begin{aligned}
\delta||u_{J2}||^p + \frac{1}{n}||u_J||^p - \frac{2}{n}||u_J||^{p-1}||u_{J2}|| &\leq (||n||_{p'} + ||h||_{p'})||\tilde{u}_J|| \qquad (3.7)\\
&\leq \tilde{k}||u_J||, \text{ for some } \tilde{k} > 0.
\end{aligned}
$$

Hence, applying (2.17) to (3.7) we have that

$$\frac{1}{2n}||u_J||^p \leq \delta||u_{J2}||^p + \frac{1}{n}||u_J||^p - \frac{3}{n}||u_J||^{p-1}||u_{J2}|| \leq \tilde{k}||u_J||. \qquad (3.8)$$

Thus,

$$\|u_J\| \leq k, \quad \text{for some} \quad k > 0. \tag{3.9}$$

However, (3.9) contradicts (3.2). Therefore, Claim 2 is established.

Continuing along with the proof of Proposition 3.5; since it is well known that $W^{m,p}(\Omega)$ is a separable reflexive Banach Space([1, page 47]) and since $mp > N$, it consequently follows from the Rellich-Kondrachov compact embedding theorem for Sobolev spaces [1, page 144] that there exists a subsequence of $\{u_J\}$ (which for ease of notation we take to be the full sequence) and a function u_n such that

$$u_n \in W^{m,p}; \tag{3.10}$$

$$\lim_{J\to\infty} \|D^\alpha u_J - D^\alpha u_n\|_p = 0, \quad \text{for} \quad |\alpha| \leq m-1; \tag{3.11}$$

$$\lim_{J\to\infty} \int_\Omega D^\alpha u_J w = \int_\Omega D^\alpha u_n w \quad \text{for all} \quad w \in L^{p'} \quad \text{and} \quad |\alpha| = m. \tag{3.12}$$

$$\lim_{J\to\infty} \int_\Omega h u_J = \int_\Omega h u_n; \tag{3.13}$$

$$\lim_{J\to\infty} \eta_{m-1}(u_J(x)) = \eta_{m-1}(u_n(x)) \quad \text{for a.e.} \quad x \in \Omega, \tag{3.14}$$

where $\eta_{m-1}(u_n(x)) = \{D^\alpha u_n(x) : |\alpha| \leq m-1\}$.

We next propose to show there exists a subsequence of $\{u_{J_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \zeta_m(u_{J_k}(x)) = \zeta_m(u_n(x)) \quad \text{for a.e.} \quad x \in \Omega. \tag{3.15}$$

where $\zeta_m(u_{J_k}(x)) = \{D^\alpha u_n(x) : |\alpha| = m\}$. To show (3.15), it is sufficient to establish the following two facts: (1) there exists a subsequence $\{u_{J_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \sum_{|\alpha|=m} (A_\alpha(x, \eta_{m-1}(u_{J_k}), \zeta_m(u_{J_k})) \quad - \quad A_\alpha(x, \eta_{m-1}(u_{J_k}), \zeta_m(u_n)))$$

$$\cdot \quad (D^\alpha u_{J_k}(x) - D^\alpha u_n(x))$$

$$= \quad 0 \quad \text{for a.e. } x \in \Omega, \tag{3.16}$$

where $\xi_m(u_{J_k}) = (\eta_{m-1}(u_{J_k}), \zeta_m(u_{J_k}))$.

(2) With $\{u_{J_k}\}_{k=1}^{\infty}$ designating the same subsequence as in (3.16),

$$\{|\zeta_m(u_{J_k}(x))|\}_{k=1}^{\infty} \quad \text{is pointwise bounded for a.e.} \quad x \in \Omega. \tag{3.17}$$

We shall soon see that both the proof of (3.16) and that of (3.17) are heavily dependent on the monotonicity assumption (A-3). The proof that (3.16) and (3.17) imply (3.15) is due to Shapiro [9, page 372]. However, we include it here for completeness. That is, there exists a finite constant $K(x)$ such that

$$|\zeta_m(u_{J_k}(x))| \leq K(x) \quad \text{for } k = 1, 2, \ldots.$$

Thus, to see that (3.16) and (3.17) imply (3.15), let Ω_1 be the subset of for which (3.14), (3.16), and (3.17) all hold simultaneously for $\{u_{J_k}\}_{k=1}^{\infty}$. Consequently

$$\text{meas}\Omega = \text{meas}\Omega_1 \tag{3.18}$$

Suppose there exists $x_o \in \Omega_1$ for which the equality in (3.15) does not hold. Hence by (3.17) there exits a further subsequence $\{\zeta_m(u_{J_{k_l}}(x_o))\}_{l=1}^{\infty}$ and a $\zeta_m^* \in R^{s_m - s_{m-1}}$ with

$$\zeta_m^* \neq \zeta_m(u_n(x_o)) \tag{3.19}$$

such that $\lim_{l\to\infty} \zeta_m(u_{J_{k_l}}(x_o)) = \zeta_m^*$. Therefore from (3.14)

$$\lim_{l\to\infty} \sum_{|\alpha|=m} \quad (A_\alpha(x_o, \eta_{m-1}(u_{J_{k_l}}), \zeta_m(u_{J_{k_l}})) - A_\alpha(x, \eta_{m-1}(u_{J_{k_l}}), \zeta_m(u_n)))$$

$$\cdot \quad (D^\alpha u_{J_{k_l}}(x_o) - D^\alpha u_n(x_o))$$

$$= \sum_{|\alpha|=m} [A_\alpha(x_o, \eta_{m-1}(u_n), \zeta_m^*) - A_\alpha(x_o, \eta_{m-1}(u_n), \zeta_m(u_n))]$$

$$(3.20) \qquad \cdot \quad [\zeta_m^* - D^\alpha u_n(x_o)].$$

From (3.19) and (A-3) we see that the right-hand side of the equality in (3.20) is strictly positive. Hence the limit on the left-hand side of the equality in (3.20) is strictly positive. However x_o is in Ω_1 and from the choice of Ω_1 and (3.16) we see that the limit on the left-hand side of the equality in (3.20) is zero. We have arrived at a contradiction. Consequently no such point like x_o exists in Ω_1. From (3.18), we have that the Lebesgue measure of Ω_1 is the same as that of Ω. We conclude that (3.15) does indeed hold once (3.16) and (3.17) are established. To establish (3.16), we shall show separately that

$$(3.21) \quad \lim_{J\to\infty} \int_\Omega \sum_{|\alpha|=m} (A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n))(D^\alpha u_J(x) - D^\alpha u_n(x)) = 0$$

and

$$(3.22) \qquad \lim_{J\to\infty} \int_\Omega \sum_{|\alpha|=m} (A_\alpha(x, \xi_m(u_J))(D^\alpha u_J(x) - D^\alpha u_n(x)) = 0.$$

The proof that (3.16) follows from (3.21) and (3.22) is again due to Shapiro [9, page 373], but we put it here for ease of reading. We observe from the difference of the above two limits that

$$\lim_{J\to\infty} \quad \int_\Omega \sum_{|\alpha|=m} [A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_J)) - A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n))]$$

$$(3.23) \qquad \cdot[D^\alpha u_J(x) - D^\alpha u_n(x)] = 0.$$

But by (A-3), the integrand in this last limit is nonnegative for a.e. $x \in \Omega$. Hence the sequence

$$\{ \sum_{|\alpha|=m} [A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_J)) \quad - \quad A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n))]$$

$$\cdot[D^\alpha u_J(x) - D^\alpha u_n(x)]\}_{J=1}^\infty$$

converges in L^1-norm to zero, and (3.16) follows immediately from Rudin[7, p. 70]. We next show that indeed (3.21) and (3.22) hold.

Equation (3.21) is also established in Shapiro [9, pages 373-374], but it is here for completeness. Observe that

$$
\begin{aligned}
&\int_\Omega A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n))[D^\alpha u_J(x) - D^\alpha u_n(x)] \\
(3.24) \qquad &= \int_\Omega [A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))] \\
&\cdot [D^\alpha u_J - D^\alpha u_n] \\
&+ \int_\Omega A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))[D^\alpha u_J(x) - D^\alpha u_n].
\end{aligned}
$$

From $u \in W^{m,p}$ and (A-2), we see that $A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n)) \in L^{p'}$ for $|\alpha| = m$. Consequently, it follows from (3.12) that the second integral on the right-hand side of the equality in (3.24) converges to zero as $k \to \infty$ for $|\alpha| = m$. Therefore (3.21) will follow once we show that

$$
\begin{aligned}
\lim_{J\to\infty} \quad &\int_\Omega [A_\alpha(x, \eta_{m-1}(u_j), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))] \\
&\cdot [D^\alpha u_J - D^\alpha u_n] = 0
\end{aligned}
$$

for $|\alpha| = m$. From (3.10) and Hölder's inequality we see that this last limit will follow once we show

$$
(3.25) \lim_{J\to\infty} \int_\Omega [A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))]^{\frac{p}{p-1}} = 0
$$

for $|\alpha| = m$. To see that (3.25) holds, we observe from (3.14) and (A-1) that the integrand in (3.25) converges to zero as $J \to \infty$ for a.e. $x \in \Omega$. Also, we see from (3.11) and (A-2) that the integrand in (3.25) is absolutely equi-integrable, i.e. given $\epsilon > 0$, there exists δ such that meas$E < \delta$ implies

$$
\int_E |A_\alpha(x, \eta_{m-1}(u_J), \zeta_m(u_n)) - A_\alpha(x, \eta_{m-1}(u_n), \zeta_m(u_n))|^{\frac{p}{p-1}} < \epsilon
$$

for $|\alpha| = m$ and $J = 1, 2, \ldots$. Consequently, we conclude from Egoroff's theorem [6], that (3.25) holds. But this establishes (3.21).

To establish (3.22), we observe from (A-2) and Claim 2 that there exists a constant $k_5 > 0$ such that

$$(3.26) \int_\Omega |A_\alpha(x, \xi_m(u_J))|^{\frac{p}{p-1}} \leq k_5, \quad \text{for} \quad 1 \leq |\alpha| \leq m \quad \text{and } J = 1, 2, \ldots.$$

Consequently, we obtain from (3.11) and Hölder's inequality that

$$\lim_{J\to\infty} \int_\Omega A_\alpha(x, \xi_m(u_J))(D^\alpha u_J(x) - D^\alpha u_n(x)) = 0 \quad \text{for} \quad 1 \leq |\alpha| \leq m-1.$$

Hence (3.22) will follow once we show

$$(3.27) \qquad \lim_{J\to\infty} \int_\Omega \sum_{1\leq|\alpha|\leq m} A_\alpha(x, \xi_m(u_J))(D^\alpha u_J(x) - D^\alpha u_n(x)) = 0.$$

To establish (3.27), we first observe from (3.10) and (2.5) that

$$(3.28) \quad \exists \{P_J u_n\}_{J=1}^\infty \text{ with } P_J u_n \in S_J \text{ s.t. } \lim_{J\to\infty} ||P_J u_n - u_n||_{W^{m,p}} = 0.$$

We therefore obtain from (3.26), (3.28), and Hölder's inequality that

$$\lim_{J\to\infty} \int_\Omega A_\alpha(x, \xi_m(u_J))(D^\alpha P_J u_n(x) - D^\alpha u_n(x)) = 0 \quad \text{for} \quad 1 \leq |\alpha| \leq m.$$

Consequently (3.27) will follow once we show

$$(3.29) \qquad \lim_{J\to\infty} \int_\Omega \sum_{1\leq|\alpha|\leq m} A_\alpha(x, \xi_m(u_J))(D^\alpha u_J(x) - D^\alpha P_J u_n(x)) = 0.$$

To establish (3.29), we invoke (2.9) and obtain that

$$\begin{aligned} Q(u_J, u_J - P_J u_n) &= \int_\Omega g^n(x, u_J)(u_J - P_J u_n) + \int_\Omega h(u_J - P_J u_n) \\ (3.30) \qquad &+ \frac{1}{n}\int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(u_J - P_J u_n). \end{aligned}$$

Next, we observe from $h \in L^{p'}$, (3.13), and (3.28) that

$$\lim_{J\to\infty} \int_\Omega h(u_J - P_J u_n) = 0. \tag{3.31}$$

Likewise, from Hölder's inequality, Claim 2, (3.11), and (3.28), we obtain

$$\frac{1}{n} \lim_{J\to\infty} \int_\Omega \operatorname{sgn}(u_J)|u_J|^{p-1}(u_J - P_J u_n) = 0. \tag{3.32}$$

Then, we see from (2.7) and (g-2) with $\epsilon = 1$ that

$$|g^n(x, u_J)| \le g_1(x) + k^{p-1} \quad \text{for a.e.} \quad x \in \Omega \quad \text{and } J = 1, 2, \ldots. \tag{3.33}$$

where $g_1 \in L^{p'}$ and k is the bound from Claim 2. Therefore, $\|g^n(x, u_J)\|_{p'}$ is bounded independent of J. Hence, to show that the first integral on the right-hand side of (3.30) converges to 0 as J goes to infinity, we re-write it as

$$\begin{aligned} \int_\Omega g^n(x,u_J)(u_J - P_J u_n) &= \int_\Omega g^n(x,u_J)(u_J - u_n) \\ &+ \int_\Omega g^n(x,u_J)(u_n - P_J u_n). \end{aligned} \tag{3.34}$$

From (3.28), we see that $\lim_{J\to\infty} \|u_n - P_J u_n\|_p = 0$. Hence, from (3.33) and by Hölder's inequality, we have that

$$\lim_{J\to\infty} \int_\Omega g^n(x,u_J)(u_n - P_J u_n) = 0. \tag{3.35}$$

Similarly, from (3.11), we have that $\|u_J - u_n\| \to 0$. This, together with (3.33) and Hölder's inequality, gives that the first integral on the right-hand side of (3.34) also converges to 0. This last fact, coupled with (3.35), says that (3.34) converges to 0 as J goes to ∞. The above fact, in conjunction with (3.30) – (3.32), gives that

$$\lim_{J\to\infty} Q(u_J, u_J - P_J u_n) = 0. \tag{3.36}$$

Next, from (2.1) we see that $Q(u_J, u_J - P_J u_n)$ is the same as the integral on the left-hand side of the equality in (3.29). Hence, the limit in (3.36) equals the limit in (3.29), and (3.29) is established. Consequently (3.22) prevails, and since (3.22) and (3.21) imply (3.16), equation (3.16) is also established. The proof that (3.17) holds is a standard argument done by Shapiro [9, pages 374-376]. One simply replaces his uniform ellipticity condition by ours, namely (A-4). Therefore, (3.15) is established.

It remains to show that Claim 2 and (3.10)-(3.15), along with the fact that $\{u_J\}_{J=1}^{\infty}$ satisfies (2.8), imply that (3.1) holds. To show this, we let $v \in \bigcup_{J=1}^{\infty} S_J$. Then it follows from (3.11) and (3.14) that

$$\lim_{k\to\infty} \int_\Omega \operatorname{sgn}(u_{J_k})|u_{J_k}|^{p-1}v = \int_\Omega \operatorname{sgn}(u_n)|u_n|^{p-1}v. \tag{3.37}$$

Also, from (g-2) with $\epsilon = 1$,

$$|g^n(x, u_J)v| \leq |vg_1| + |v|k^{p-1} \quad \text{for a.e.} \quad x \in \Omega, \tag{3.38}$$

where $g_1 \in L^{p'}$ and k is the bound from Claim 2. Thus, we see that $|vg_1| \in L^1(\Omega)$. Hence we conclude from the Lebesgue dominated convergence theorem, (g-1), (3.14), and (3.38) that

$$\lim_{k\to\infty} \int_\Omega g^n(x, u_{J_k})v = \int_\Omega g^n(x, u_n)v. \tag{3.39}$$

Next, we see from (A-2) in conjunction with Claim 2 and Hölder's inequality that

$$\{A_\alpha(x, \xi_m(u_{J_k}))D^\alpha v\}_{k=1}^{\infty} \quad \text{is uniformly equi-integrable} \tag{3.40}$$

for $1 \leq |\alpha| \leq m$. Also, (A-1) along with (3.14) and (3.15) yields

$$\lim_{k\to\infty} A_\alpha(x, \xi_m(u_{J_k}))D^\alpha v(x) = A_\alpha(x, \xi_m(u_n))D^\alpha v(x)$$

for a.e. $x \in \Omega$ and $1 \leq |\alpha| \leq m$. This fact along with (3.40), (2.1) and Egoroff's theorem gives $\lim_{k\to\infty} Q(u_{J_k}, v) = Q(u_n, v)$. From Proposition 2.4, (3.37), (3.39), and this last limit, we have that

$$(3.41) \quad Q(u_n, v) - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_n)|u_n|^{p-1}v = \int_\Omega hv \ + \ \int_\Omega g^n(x, u_n)v, \quad \forall v \in \bigcup_{J=1}^{\infty} S_J.$$

It is a straight forward density argument to conclude that (3.41) holds also for all $v \in W^{m,p}(\Omega)$. Hence, (3.1) is established, and the proof of Proposition 3.5 is complete. ∎

4. Resonance $W^{m,p}(\Omega)$. In this section, we prove Theorem 2.1 which allows for g to grow superlinearly under the restriction that $mp > N$.

Proof of Theorem 2.1:

Employing the familiar Galerkin approximation scheme, we first invoke Proposition 3.5 and obtain a sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$(4.1) \qquad u_n \in W^{m,p} \quad \text{satisfies} \quad (3.1) \text{ for } n = 1, 2, \ldots.$$

We claim that

$$(4.2) \qquad \{||u_n||_{W^{m,p}}\}_{n=1}^{\infty} \quad \text{is uniformly bounded.}$$

Suppose claim (4.2) is false. Then without lost of generality, we can assume that

$$(4.3) \qquad \lim_{n\to\infty} ||u_n||_{W^{m,p}} = \infty.$$

Next, we let $\tilde{u}_n$ be as defined by (3.3) except that we replace J by n everywhere. Thus, letting v be $\tilde{u}_n$ in (3.1) and applying (g-2), we obtain,

$$Q(u_n, \tilde{u}_n) \ = \ \frac{1}{n}\int_\Omega sgn(u_n)|u_n|^{p-1}(\tilde{u}_n) + \int_\Omega g^n(x, u_n)\tilde{u}_n + \int_\Omega h\tilde{u}_n$$

$$\leq \frac{1}{n}||u_n||^{p-1}||\tilde{u}_n|| + \epsilon||u_n||^{p-1}||\tilde{u}_n||$$

$$(4.4) \qquad + \quad ||g_\epsilon||_{p'}||\tilde{u}_n|| + ||h||_{p'}||\tilde{u}_n||.$$

By the definition of $\tilde{u}_n$ and applying arguments similar to those used between (3.6) and (3.7), we obtain $||\tilde{u}_n|| \leq k||u_n||$, for some $k > 0$. Therefore, it follows that $||g_\epsilon||_{p'}||\tilde{u}_n|| + ||h||_{p'}||\tilde{u}_n|| \leq \tilde{k}||u_n||$, where $\tilde{k} = k(||g_\epsilon||_{p'} + ||h||_{p'})$. Next, using (A-4), the fact that $Q(u_n, \tilde{u}_n) = Q(u_n, u_n)$, in conjunction with (4.4), we obtain

$$c_o \int_\Omega \{ \sum_{1 \leq |\alpha| \leq m} |D^\alpha u_n|^2 \}^{\frac{p}{2}} \leq \frac{k}{n}||u_n||^p + k\epsilon||u_n||^p + \tilde{k}||u_n||$$

(4.5) for some $\tilde{k} > 0$.

Next setting

$$(4.6) \qquad v_n = \frac{u_n}{||u_n||_{W^{m,p}}},$$

and dividing both side of (4.5) by $||u_n||^p_{W^{m,p}}$, and since $||u_n||_{W^{m,p}} \to \infty$ and $\epsilon > 0$ is arbitrary, we have

$$(4.7) \qquad \lim_{n \to \infty} c_o \int_\Omega \{ \sum_{1 \leq |\alpha| \leq m} |D^\alpha v_n|^2 \}^{\frac{p}{2}} = 0.$$

Thus, since $c_o > 0$, we have that

$$(4.8) \qquad \lim_{n \to \infty} \sum_{1 \leq |\alpha| \leq m} ||D^\alpha v_n||^p = 0.$$

From (4.6) we see that $||v_n||_{W^{m,p}} = 1$ for $n = 1, 2, ...$

$$(4.9) \qquad \text{Hence,} \quad ||v_n||^p_{W^{m,p}} = 1.$$

Hence, since $1 = ||v_n||^p_{W^{m,p}} = ||v_n||^p_p + \sum_{1 \leq |\alpha| \leq m} ||D^\alpha v_n||^p_p$, we infer from (4.8) that

$$(4.10) \qquad \lim_{n \to \infty} ||v_n||_p = 1.$$

Clearly, $\{||v_n||^p_{W^{m,p}}\}_{n=1}^{\infty}$ is a uniformly bounded sequence, thus there exists a subsequence and a function v_o with the following properties:

$$v_n \to v_o \in W^{m,p}(\Omega), \text{ weakly;} \tag{4.11}$$

$$\lim_{n\to\infty} ||D^\alpha v_n - D^\alpha v_o||_p = 0, \quad \text{for} \quad |\alpha| \le m-1; \tag{4.12}$$

$$\lim_{n\to\infty} \int_\Omega D^\alpha v_n w = \int_\Omega D^\alpha v_o w \quad \text{for all} \quad w \in L^{p'} \quad \text{and} \quad |\alpha| = m. \tag{4.13}$$

$$\lim_{n\to\infty} \int_\Omega h v_n = \int_\Omega h v_o, \quad \text{since} \quad h \in L^{p'}. \tag{4.14}$$

$$\lim_{n\to\infty} D^\alpha v_n(x) = D^\alpha v_o(x) \quad \text{for a.e.} \quad x \in \Omega \quad \text{and} \quad |\alpha| \le m-1. \tag{4.15}$$

Next, from (4.8) and (4.13), and Hölder's Inequality we have

$$\int_\Omega D^\alpha v_o w = 0 \quad \text{for} \quad w \in L^{p'}, \quad 1 \le |\alpha| \le m.$$

Consequently, $D^\alpha v_o = 0$ a.e. in Ω for $1 \le |\alpha| \le m$. Since Ω is a bounded open connected set meeting the cone property, we conclude that $v_o =$ constant a.e. in Ω. From (4.10) and (4.12) we obtain that

$$||v_o||_p = 1.$$

Hence, this constant is a nonzero either positive or negative quantity. We shall assume that it is positive. Since a similar argument prevails for the case when the constant is negative. Let

$$v_o = c_4 \quad \text{a.e.} \quad x \in \Omega, \quad c_4 = (\text{meas}\Omega)^{-\frac{1}{p}}. \tag{4.16}$$

Next, we invoke (4.1) with $v = v_o = c_4 > 0$ a.e. $x \in \Omega$ to obtain,

$$Q(u_n, v_o) - \frac{1}{n} \int_\Omega \text{sgn}(u_n)|u_n|^{p-1} v_o = \int_\Omega g^n(x, u_n) v_o + \int_\Omega h v_o. \tag{4.17}$$

From the definition of Q we have that $Q(u_n, v_o) = 0$. Therefore,

$$\int_\Omega g^n(x, u_n)v_o + \int_\Omega hv_o \leq 0.$$

Since by hypothesis $mp > N$, we have from the compact embedding theorems that

$$\lim_{n\to\infty} v_n = v_o \quad \text{uniformly.} \tag{4.18}$$

Applying (g-3) and using (4.18), we have for $n \geq n_o$ that

$$g^n(x, u_n) \geq -c(x) - \frac{d(x)}{|u_n|}.$$

Therefore, for $n \geq n_o$ and since $v_o > 0$ a.e., the following holds

$$g^n(x, u_n)v_o \geq -v_o c(x) - d(x)\frac{v_o}{|u_n|}.$$

Thus, we can apply Fatou's lemma to the following quantity,

$$\begin{aligned}&\liminf_{n\to\infty}\Big(\int_\Omega (g^n(x, u_n)v_o + c(x)v_o + d(x)\frac{v_o}{|u_n|})\\ &- \int_\Omega (c(x)v_o + d(x)\frac{v_o}{|u_n|})\Big) \leq -\int_\Omega hv_o.\end{aligned}$$

Now, since $u_n = v_n ||u_n||_{W^{m,p}} \to +\infty$, we obtain

$$\begin{aligned}\int_\Omega \liminf_{n\to\infty} g^n(x, u_n) \quad &+ \quad \int_\Omega \liminf_{n\to\infty}(c(x)v_o + d(x)\frac{v_o}{|u_n|})\\ &- \quad \liminf_{n\to\infty} \int_\Omega (c(x)v_o + d(x)\frac{v_o}{|u_n|})) \leq -\int_\Omega hv_o.\end{aligned}$$

However,

$$\liminf_{n\to+\infty} g^n(x, u_n) \geq g_+(x).$$

Therefore,

$$\int_\Omega g_+(x)v_o \leq -\int_\Omega hv_o.$$

But this yields a contradiction to the Landesman-Lazer conditions (2.2). Hence, we cannot have (4.3) holding. Thus there exits a constant $K_6 > 0$ such that

(4.19) $$||u_n||_{W^{m,p}} \leq K_6, \quad \text{for } n = 1, 2, \ldots.$$

As before, there exists a subsequence of $\{u_n\}$ (which for ease of notation we take to be the full sequence) and a function u such that

(4.20) $$u_n \to u \in W^{m,p}, \text{ weakly;}$$

(4.21) $$\lim_{n\to\infty} ||D^\alpha u_n - D^\alpha u||_p = 0, \quad \text{for} \quad |\alpha| \leq m-1;$$

(4.22) $$\lim_{n\to\infty} \int_\Omega D^\alpha u_n w = \int_\Omega D^\alpha u w \quad \text{for all} \quad w \in L^{p'} \quad \text{and} \quad |\alpha| = m.$$

(4.23) $$\lim_{n\to\infty} \int_\Omega h u_n = \int_\Omega h u;$$

(4.24) $$\lim_{n\to\infty} \eta_{m-1}(u_n(x)) = \eta_{m-1}(u(x)) \quad \text{for a.e.} \quad x \in \Omega,$$

where $\eta_{m-1}(u_n(x)) = \{D^\alpha u_n(x) : |\alpha| \leq m-1\}$.
We next propose to show there exists a subsequence of $\{u_{n_k}\}_{k=1}^\infty$ such that

(4.25) $$\lim_{k\to\infty} \zeta_m(u_{n_k}(x)) = \zeta_m(u(x)) \quad \text{for a.e.} \quad x \in \Omega,$$

where $\zeta_m(u_{n_k}(x)) = \{D^\alpha u(x) : |\alpha| = m\}$. As in the proof of Proposition 3.5, once (4.25) is established, it will be an easy matter to establish Theorem 2.1 from (4.19) – (4.25). However, the proof that (4.25) holds, is parallel to the proof of (3.15). One simply replaces u_J by u_n, u_{J_k} by u_{n_k}, and u_n by u.

To complete the proof of the theorem, we have to show that (4.19) – (4.25) along with (4.1) gives (2.3). In order to accomplish this, let $v \in W^{m,p}(\Omega)$ be given. Then it follows from (4.1), (3.1), and (2.1) that

$$\sum_{1\le|\alpha|\le m}\int_\Omega A_\alpha(x,\xi_m(u_{n_k}))D^\alpha v - \frac{1}{n}\int_\Omega \operatorname{sgn}(u_{n_k})|u_{n_k}|^{p-1}v$$

$$(4.26) \qquad = \int_\Omega g^{n_k}(x,u_{n_k})v+\int_\Omega hv.$$

From (4.19), we see that $\|u_{n_k}\|_p \le K_6$ for k=1,2,.... Hence it follows from Hölder's inequality and $v\in W^{m,p}$ that

$$(4.27) \qquad \lim_{k\to\infty}\frac{1}{n_k}\int_\Omega \operatorname{sgn}(u_{n_k})|u_{n_k}|^{p-1}v=0.$$

Next, from (g-2) with $\epsilon = 1$, we see that

$$(4.28) \qquad |g^n(x,u_n)|\le g_1(x)+|u_n|^{p-1}, \quad \text{for k=1,2,...}$$

where $g_1\in L^{p'}$. Also, we see from Hölder's inequality that

$$(4.29) \qquad \int_E |u_n|^{p-1}|v| \le \{\int_E |u_n|^p\}^{\frac{p-1}{p}}\{\int_E |v|^p\}^{\frac{1}{p}},$$

where E is a measurable subset of Ω. From Claim 2 and (4.19), we see that the first integral on the right-hand side of the inequality in (4.29) is uniformly bounded in n. Hence it follows from (4.28) and (4.29) that

$$(4.30) \qquad \{g^n(x,u_n)v\}_{n=1}^\infty \quad \text{is absolutely equi-integrable,}$$

and from (g-1), (2.7), and (4.24) we have that

$$(4.31) \qquad \lim_{n\to\infty} g^n(x,u_n)v(x)=g(x,u)v(x) \quad \text{a.e. in } \Omega.$$

Therefore, from (4.30), (4.31) and by Vitali's theorem we have that

$$(4.32) \qquad \lim_{k\to\infty}\int_\Omega g^{n_k}(x,u_{n_k})v=\int_\Omega g(x,u)v.$$

Next, with $\{u_{n_k}\}_{k=1}^{\infty}$ the subsequence given in (4.25), we obtain from (A-1), (4.24), and (4.25) that

$$(4.33)\lim_{k\to\infty} A_\alpha(x, \xi_m(u_{n_k}(x)))D^\alpha v(x) = A_\alpha(x, \xi_m(u(x)))D^\alpha v(x), \text{ a.e. in } \Omega,$$

for $1 \leq |\alpha| \leq m$. Also, we see from (4.19), (A-2), and Hölder's inequality that

$$(4.34) \qquad \{A_\alpha(x, \xi_m(u_{n_k}(x))D^\alpha v\}_{n=1}^{\infty} \quad \text{is absolutely equi-integrable}$$

for $1 \leq |\alpha| \leq m$. Hence, it follows from (4.33), (4.34), and Vitali's theorem that

$$(4.35) \qquad \lim_{k\to\infty} \int_\Omega A_\alpha(x, \xi_m(u_{n_k}(x)))D^\alpha v = \int_\Omega A_\alpha(x, \xi_m(u))D^\alpha v,$$

for $1 \leq |\alpha| \leq m$. From (4.26), (4.27), (4.35), and (4.35), we obtain that

$$\sum_{1\leq|\alpha|\leq m} \int_\Omega A_\alpha(x, \xi_m(u))D^\alpha v = \int_\Omega g(x, u)v + \int_\Omega hv, \quad \forall v \in W^{m,p}(\Omega).$$

But from (2.1), we see that this last equality is the same as (2.3), and the proof of Theorem 2.1 is complete. ■

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